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Transportation matrices with staircase patterns and majorization

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Abstract

We consider some questions concerning transportation matrices with a certain nonzero pattern. For a given staircase pattern we characterize those row sum vectors R and column sum vectors S such that the corresponding class of transportation matrices with the given row and column sums and the given pattern is nonempty. Two versions of this problem are considered. Algorithms for finding matrices in these matrix classes are introduced and, finally, a connection to the notion of majorization is discussed.

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1. Introduction

Let $R = (r_1, r_2, \dots, r_m)$ and $S = (s_1, s_2, \dots, s_n)$ be positive vectors of length m and n, respectively, such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Throughout this paper R and S are of this assumed form.

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Define

$$\mathcal{N}(R, S) = \text{the set of all nonnegative } m \times n \text{ matrices } A \text{ with row sum vector } R \text{ and column sum vector } S.$$

So, if $A \in \mathcal{N}(R, S)$, then $\sum_{j=1}^{n} a_{ij} = r_i$ $(i \leqslant m)$ and $\sum_{i=1}^{m} a_{ij} = s_j$ $(j \leqslant n)$. The matrix class $\mathcal{N}(R, S)$ is a bounded polyhedron, and therefore a polytope, in the vector space of all real $m \times n$ matrices. It is known as the *transportation polytope* as its matrices correspond to transportation plans for shipping material from m sources to n destinations with given amounts in each of these sources and destinations. For a survey of known results concerning the matrix class $\mathcal{N}(R, S)$ we refer to the recent book [4], or to [11]. Further discussion of this class from an optimization point of view, represented by the *transportation problem*, is found in most textbooks on linear programming, network flows or operations research, see e.g. [10,7,1]. An interesting presentation of the history of the transportation problem (and the maximum flow problem) may be found in [9].

The *pattern* $\mathscr{P}(A)$ of a nonnegative matrix A is the set of positions of its positive entries, i.e., $\mathscr{P}(A) = \{(i, j) : a_{ij} > 0\}$. The pattern $\mathscr{P}(A)$ may be identified with a (0,1)-matrix W of size $m \times n$ where the ones in W are in the positions specified by $\mathscr{P}(A)$. Patterns of transportation matrices were characterized in [2] (see the theorem below); for this and related results see [4]. The patterns of real (not necessarily nonnegative) matrices with given line sums were characterized in [6]. The patterns of the *vertices* of the transportation polytope correspond to forests (spanning subgraphs with no cycles) in the bipartite graph associated with a $m \times n$ matrix; for this and other results concerning transportation polytopes, see [4].

If A is an $m \times n$ matrix and $I \subseteq \{1, 2, \ldots, m\}$, $J \subseteq \{1, 2, \ldots, n\}$, we let $A[I, J)$ denote the submatrix of A with rows in I and columns in $\{1, 2, \ldots, n\} \setminus J$. The submatrix $A(I, J]$ is defined similarly. An all zero matrix is denoted by O (we write O_{pq} if the size $p \times q$ needs to be indicated). Similarly, an all ones matrix is denoted by J (or J_{pq} to indicate its size). The symmetric difference of two sets U and V is denoted by $U \triangle V$ (which is equal to $(U \cup V) \setminus (U \cap V)$).

The following theorem from [2] (see also [4]) contains a basic result concerning $\mathcal{N}(R, S)$; it characterizes the pattern of transportation matrices.

Theorem 1 [2]. *Let W be an $m \times n$ $(0, 1)$-matrix. Then $\mathcal{N}(R, S)$ contains a matrix A with pattern W if and only if the following condition holds*:

$$\sum_{l \in L} s_l \geqslant \sum_{k \in K} r_k \tag{1}$$

for each $\emptyset \subset K \subset \{1, 2, \ldots, m\}$ and $\emptyset \subset L \subset \{1, 2, \ldots, n\}$ satisfying $W[K, L) = O$, and where equality holds in (1) if and only if $W(K, L] = O$.

The condition in (1) is called the (R, S)-*support condition*. It is possible to derive this theorem from network flow theory (which is presented in e.g. [1,5]).

The goal in this paper is to investigate transportation matrices with staircase patterns, as defined in Section 2. The main results are in Section 3 where we show that a much smaller system of linear inequalities than the one given by the (R, S)-support condition characterizes the existence of a transportation matrix with a fixed staircase pattern. This result generalizes a theorem in [11] (in Chapter 7, on truncated transportation polytopes). Moreover, in the final section we discuss a connection to the notion of majorization.

2. Staircase patterns

In this section we discuss staircase patterns and some basic results for these structures.

Consider integers

$$1 = v_1 \leqslant v_2 \leqslant \cdots \leqslant v_m \quad \text{and} \quad h_1 \leqslant h_2 \leqslant \cdots \leqslant h_m = n,$$

such that $v_i \leqslant h_i$ $(i \leqslant m)$ and $v_i \leqslant h_{i-1} + 1$ $(2 \leqslant i \leqslant m)$. Define the associated (0, 1)-matrix $W = [w_{ij}]$ by $w_{ij} = 1$ if $v_i \leqslant j \leqslant h_i$ and $w_{ij} = 0$ otherwise $(i \leqslant m)$. We call W and its pattern $\mathscr{P}(W)$ a *staircase matrix* and a *staircase pattern*, respectively. Every line (row or column) in W has the form

$$0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0,$$

where the sequence of consecutive ones is nonempty.

For such a staircase matrix $W = [w_{ij}]$ we define

$$C_W^1 = \{(p, q) : w_{pq} = 1,\ w_{p,q+1} = 0,\ w_{p+1,q+1} = 1\},$$

$$C_W^2 = \{(p, q) : w_{pq} = 1,\ w_{p+1,q} = 0,\ w_{p+1,q+1} = 1\}.$$

Each $(p, q) \in C_W^1 \cup C_W^2$ will be called a *critical position*. Clearly, $|C_W^1|, |C_W^2| \leqslant \min\{m, n\} - 1$. Note that if $(p, q) \in C_W^1$, then W has the form

$$W = \begin{bmatrix} W_{11} & O_{p,n-q} \\ W_{21} & W_{22} \end{bmatrix} \tag{2}$$

and if $(p, q) \in C_W^2$, then W has the form

$$W = \begin{bmatrix} W_{11} & W_{12} \\ O_{m-p,q} & W_{22} \end{bmatrix}. \tag{3}$$

In both cases the submatrix W_{11} is of size $p \times q$ and the entry of W_{22} in position (1,1) is 1. If $(p, q) \in C_W^1 \cap C_W^2$, then W is the direct sum of two staircase matrices of size $p \times q$ and $(m - p) \times (n - q)$, respectively. The pair (C_W^1, C_W^2) determines the staircase matrix uniquely.

Example 1. Let $m = 5$, $n = 7$, $v_1 = v_2 = 1$, $v_3 = v_4 = 3$, $v_5 = 5$, and $h_1 = h_2 = 3$, $h_3 = 5$, $h_4 = 6$, $h_5 = 7$. The corresponding staircase matrix is

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The critical positions are $C_W^1 = \{(2, 3), (3, 5), (4, 6)\}$ and $C_W^2 = \{(2, 2), (4, 4)\}$.

The following simple lemma will be useful later.

Lemma 2. *Let W be a staircase matrix of size $m \times n$. Then either W is a direct sum of an all ones matrix and a smaller staircase matrix (possibly vacuous), or there are integers $1 \leqslant p < m$, $1 \leqslant q < n$ such that W is of one of the following two types, denoted by* T_1 *and* T_2*:*

$$(\mathrm{T}_1) \begin{bmatrix} J_{pq} & W_{12} \\ O_{m-p,q} & W_{22} \end{bmatrix}, \quad (\mathrm{T}_2) \begin{bmatrix} J_{pq} & O_{p,n-q} \\ W_{21} & W_{22} \end{bmatrix},$$

where, in case T_1*, the first column of the submatrix* W_{12} *contains only ones, and, in case* T_2*, the first row of the submatrix* W_{21} *contains only ones.*

Proof. Let U and V denote the first column and the first row in W; note that the first component in these vectors is 1. Assume $W \neq J_{m,n}$. Then either U or V contains a zero. If only U (resp. V) contains a zero, then W must be of type T_1 (resp. T_2). Otherwise, both U and V contain a zero, say that U consists of p ones followed by zeros, and V contains q ones followed by zeros. Now, depending on the number of leading zeros in row $p+1$ and column $q+1$ of W it is easy to see that W is either a direct sum of an all ones matrix of size $p \times q$ and a smaller staircase matrix, or one of the two cases T_1 or T_2 must occur. □

We now consider the term rank of a staircase matrix. Recall that the *term rank* of a (0,1)-matrix W is the maximum number of 1's in the matrix no two of which lie in the same line. The term rank of W is denoted by $\rho(W)$.

Consider the following algorithm. By a *permitted position* we mean a position in the support of the given staircase matrix W. Here, in the rest of this section, it is convenient to extend the definition of staircase pattern by allowing leading rows and/or columns of all zeros.

Algorithm TR

1. Initialize the $m \times n$ matrix A by putting a 0 in every non-permitted position. Let $\widetilde{A} = A$.
2. Let i be smallest possible such that row i of $\widetilde{A}$ contains a permitted position, and let j be smallest possible such that position (i, j) is permitted. Put a 1 in the position in A corresponding to (i, j).
3. Update $\widetilde{A}$ by striking out the i leading rows and the j leading columns.
4. If $\widetilde{A}$ is the empty matrix, stop, and fill in the remaining entries in A with zeros. Otherwise, go to Step 2.

This simple algorithm has complexity $O(\min\{m, n\})$.

Theorem 3. *Let* W *be a staircase matrix of size* $m \times n$*. Then Algorithm TR finds the term rank* $\rho(W)$ *of* W*, and a corresponding set of* $\rho(W)$ 1*'s in* W *no two of which are in the same line.*

Proof. A (0, 1)-matrix of size $m \times n$ with $\rho(W)$ 1's, all in permitted positions and with no two 1's in the same line, will be called a *maximizer*. We now prove the following:

Claim 1. *There is a maximizer with a* 1 *in position* (i, j) *where* i *is smallest possible such that row* i *in* W *is nonzero, and* j *is smallest possible such that* $w_{ij} = 1$.

Proof of Claim. Let A be a maximizer and consider the position (i, j) as in the claim. If $a_{ij} = 1$ we are done, so assume that $a_{ij} = 0$. If row i of A contains a 1, but column j does not, we move the 1 in row i to position (i, j) and thereby obtain another maximizer with the desired property. Note that (i, j) is a permitted position due to the staircase pattern. Similarly, if column j of A contains a 1, but row i does not, we move the 1 in column j to (the permitted) position (i, j) and thereby obtain another maximizer as desired. Finally, consider the case when A has a 1 in row i and in column j, say $a_{ik} = 1$, $a_{uj} = 1$ where $k > j$ and $u > i$. Then $a_{ij} = a_{uk} = 0$, and by an interchange applied to the 2×2 submatrix consisting of rows i and u, and columns j and k, we obtain a maximizer with $a_{ij} = 1$. (Such an interchange complements the entries in positions (i, j), (i, k), (u, j), (u, k)). Again, the staircase structure implies that (i, j) and (u, k) are permitted positions. This proves the claim.

It follows from the claim that putting a 1 in indicated position (i, j) does not prevent us in finding a maximizer. This is what Algorithm TR does first. Then we can remove the i leading rows and the j leading columns since they cannot contain a 1, and the correctness of Algorithm TR now follows by induction. □

3. The main results

Let W be a given staircase matrix of size $m \times n$. Define

$$\mathcal{N}_W(R, S) = \{A \in \mathcal{N}(R, S) : \mathcal{P}(A) = \mathcal{P}(W)\}$$

as the set of transportation matrices with pattern given by W. Similarly, let

$$\mathcal{N}_{\leqslant W}(R, S) = \{A \in \mathcal{N}(R, S) : \mathcal{P}(A) \subseteq \mathcal{P}(W)\}$$

be the set of transportation matrices A with pattern contained in the pattern of W, i.e., the positive entries in A are in (some of) the positions of the ones in W.

The set $\mathcal{N}_{\leqslant W}(R, S)$ is a polytope. Actually, it is a face of the transportation polytope $\mathcal{N}(R, S)$; this face is obtained by setting all entries outside the pattern of W to zero. On the other hand, the set $\mathcal{N}_W(R, S)$ may not even be convex. A thorough discussion of similar sets for more general patterns W (other than staircase patterns) is found in [4,11]. We now strengthen some of these results when W is a staircase matrix.

The main goal is to characterize whenever the sets $\mathcal{N}_{\leqslant W}(R, S)$ and $\mathcal{N}_W(R, S)$ are nonempty, and to give efficient methods for finding matrices in these classes. These characterizations are given in terms of a linear system of inequalities involving R and S. There is one such inequality for every critical position in W. We remark that the necessity of the conditions is not hard to see, but the sufficiency requires some more work.

3.1. *The class* $\mathcal{N}_{\leqslant W}(R, S)$

The following theorem characterizes when the class $\mathcal{N}_{\leqslant W}(R, S)$ is nonempty. The characterization involves the critical positions as defined in Section 2.

Theorem 4. *Let W be a staircase matrix of size $m \times n$. Then $\mathcal{N}(R, S)$ contains a matrix A with $\mathcal{P}(A) \leqslant \mathcal{P}(W)$ if and only if R and S satisfy*

$$\begin{aligned} &\sum_{i=1}^{p} r_i \leqslant \sum_{j=1}^{q} s_j \quad ((p, q) \in C_W^1), \text{ and} \\ &\sum_{i=1}^{p} r_i \geqslant \sum_{j=1}^{q} s_j \quad ((p, q) \in C_W^2). \end{aligned} \tag{4}$$

The proof of Theorem 4 is given below. We call the inequalities in (4) the *staircase support condition*. Note that if $(p, q) \in C_W^1 \cap C_W^2$, then W decomposes into a direct sum of smaller staircase matrices, and the staircase support condition contains the equation $\sum_{i=1}^{p} r_i = \sum_{j=1}^{q} s_j$.

Example 1 cont. The staircase support condition (4) for W as in Example 1 is the linear system

$$\begin{aligned}
&r_1 + r_2 \leqslant s_1 + s_2 + s_3,\\
&r_1 + r_2 + r_3 \leqslant s_1 + s_2 + s_3 + s_4 + s_5,\\
&\underline{r_1 + r_2 + r_3 + r_4 \leqslant s_1 + s_2 + s_3 + s_4 + s_5 + s_6,}\\
&r_1 + r_2 \geqslant s_1 + s_2,\\
&r_1 + r_2 + r_3 + r_4 \geqslant s_1 + s_2 + s_3 + s_4.
\end{aligned}$$

For instance, $R = (4, 6, 3, 2, 3)$ and $S = (4, 2, 5, 3, 2, 1.4, 0.6)$ satisfy these inequalities and a matrix in $\mathcal{N}_{\leqslant W}(R, S)$ is

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1.4 & 0.6 \end{bmatrix}. \qquad \square$$

In order to prove Theorem 4 we discuss the following algorithm. It is sometimes called the *North-West Corner rule* (see e.g. [7,4]) and it is often used to construct a feasible solution in the classical transportation problem, i.e., it constructs a matrix in $A = [a_{ij}] \in \mathcal{N}(R, S)$. It starts in the upper left (North-West) corner and constructs the first row or column as a multiple of the first unit vector; the remaining matrix is constructed recursively.

Algorithm NWC

1. (i) If $r_1 \leqslant s_1$, let $a_{11} = r_1$ and $a_{12} = \cdots = a_{1n} = 0$. The first row of A has been constructed; update s_1 by subtracting r_1 and proceed to Step 2 with the updated S and row sum vector R' which is obtained from R by deleting the first component.
   (ii) If $r_1 > s_1$, let $a_{11} = s_1$ and $a_{21} = \cdots = a_{m1} = 0$. Then the first column of A has been constructed; update r_1 by subtracting s_1 and proceed to Step 2 with the updated R and column sum vector S' which is obtained from S by deleting the first component.
2. Repeat Step 1 until the remaining submatrix is empty.

We now use Algorithm NWC to prove the previous theorem; the central point is to prove that the constructed matrix A has support contained in the desired staircase pattern.

Proof of Theorem 4. To prove the necessity of the conditions assume that $A \in \mathcal{N}_{\leqslant W}(R, S)$. Let $(p, q) \in C^1_W$. Then, using (2) we get

$$\begin{aligned}
\sum_{j=1}^{q} s_j &= \sum_{j=1}^{q}\sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{q} a_{ij}\\
&\geqslant \sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij} = \sum_{i=1}^{p}\sum_{j=1}^{n} a_{ij}\\
&= \sum_{i=1}^{p} r_i.
\end{aligned}$$

The corresponding inequality for $(p, q) \in C^2_W$ is proved similarly (from (3)), which proves the necessity of the condition (4).

We next prove the sufficiency of this condition. A straightforward induction argument shows that it is sufficient to consider the case when W is not the direct sum of smaller staircase matrices. In this case $C_W^1 \cap C_W^2$ is empty. So assume that (4) holds. Apply Algorithm NWC to the pair (R, S), and let A be the resulting matrix.

Claim 2. *$\mathscr{P}(A) \subseteq \mathscr{P}(W)$, i.e., $A \in \mathscr{N}_{\leqslant W}(R, S)$.*

Proof of Claim. Assume that the claim is not true, and let (u, v) be the *first* position outside $\mathscr{P}(W)$ in which the algorithm sets the entry to some positive number. There are two possible cases.

Case 1: $w_{1v} = \cdots = w_{uv} = 0$. Then there is a critical element $(p, q) \in C_W^1$ with $p \geqslant u$ and $q = v - 1$. Moreover, since Algorithm NWC places the positive entries along a path of increasing row and column numbers, we must have

$$a_{ij} = 0 \quad \text{for all } (i, j) \text{ with } i > u \text{ and } j \leqslant q.$$

This implies

$$\sum_{i=1}^{p} r_i > \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{q} a_{ij} = \sum_{j=1}^{q} s_j.$$

But this contradicts (4).

Case 2: $w_{u1} = \cdots = w_{uv} = 0$. Then there is a critical element $(p, q) \in C_W^2$ with $p = u - 1$ and $q \geqslant v$, and we can argue as above to obtain an inequality which contradicts (4).

This proves the Claim, and therefore the class $\mathscr{N}_{\leqslant W}(R, S)$ is nonempty. $\square$

As seen from the above proof we may use Algorithm NWC to check if $\mathscr{N}_{\leqslant W}(R, S)$ is nonempty and, if so, find a matrix in $\mathscr{N}_{\leqslant W}(R, S)$. Thus, the algorithm indirectly tests the staircase support condition (4).

It is easy to verify that the matrix A determined by Algorithm NWC is a vertex of the polytope $\mathscr{N}_{\leqslant W}(R, S)$. Actually, *any* vertex of this polytope may be produced by an algorithm similar to Algorithm NWC where, in each stage, a position (i, j) is selected and the corresponding entry is set to $\min\{r_i, s_j\}$; see [4] for a similar result for $\mathscr{N}(R, S)$.

3.2. The class $\mathscr{N}_W(R, S)$

We turn the attention to the class $\mathscr{N}_W(R, S)$. The main result in this subsection is the following theorem.

Theorem 5. *Let W be a staircase matrix of size $m \times n$. Then $\mathscr{N}(R, S)$ contains a matrix A with $\mathscr{P}(A) = \mathscr{P}(W)$ if and only if (4) holds and, in addition, these inequalities are strict for all $(p, q) \in C_W^1 \triangle C_W^2$.*

We shall give an algorithmic proof of this theorem, and to this end we consider the following algorithm. It constructs, under the conditions in Theorem 5, a matrix A in the class $\mathscr{N}_W(R, S)$. In the algorithm we use the different types of W discussed in Lemma 2. The idea is to construct a matrix $A \in \mathscr{N}_W(R, S)$ in several stages where, in each stage, a set of leading rows and/or columns is constructed. The entries are chosen in a certain “proportional manner” by adding a certain rank one matrix.

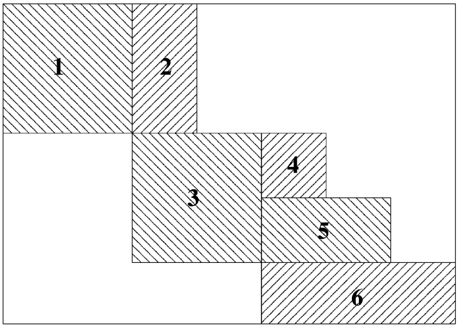

Fig. 1. The blocks determined by Algorithm A.

Algorithm A

1. Initialize: let $\widetilde{W} = W$, $\widetilde{R} = R$, and $\widetilde{S} = S$.
2. Construct a block in A as follows:

   (i) If $\widetilde{W}$ is of type T_1, let $\tilde{\tau} = \sum_{i \leqslant p} \tilde{r}_i$ and define $a_{ij} = \tilde{r}_i \tilde{s}_j / \tilde{\tau}$ ($i \leqslant p$, $j \leqslant q$), and $a_{ij} = 0$ ($i > p$, $j \leqslant q$). Go to Step 3.
   (ii) If $\widetilde{W}$ is of type T_2, let $\tilde{\tau} = \sum_{j \leqslant q} \tilde{s}_j$ and define $a_{ij} = \tilde{r}_i \tilde{s}_j / \tilde{\tau}$ ($i \leqslant p$, $j \leqslant q$), and $a_{ij} = 0$ ($i \leqslant p$, $j > q$). Go to Step 3.
   (iii) If $\widetilde{W} = J_{p,q} \oplus W'$ where $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$, let $\tilde{\tau} = \sum_{i \leqslant p} \tilde{r}_i = \sum_{j \leqslant q} \tilde{s}_j$ and define $a_{ij} = \tilde{r}_i \tilde{s}_j / \tilde{\tau}$ ($i \leqslant p$, $j \leqslant q$), and $a_{ij} = 0$ when $i \leqslant p$, $j > q$ or $i > p$, $j \leqslant q$.

3. If all entries of A have been determined, terminate. Otherwise, let $\widetilde{W}$ be the submatrix of W corresponding to the remaining positions in A, and update the row sum vector $\widetilde{R}$ and the column sum vector $\widetilde{S}$ according to the block in A that was determined. Go to Step 2.

Example 1. cont. Consider again $R = (4, 6, 3, 2, 3)$ and $S = (4, 2, 5, 3, 2, 1.4, 0.6)$. These vectors satisfy the condition of Theorem 5. Algorithm A requires six stages (iterations) and these determine, respectively, leading rows or columns as follows: two columns, two rows, two columns, one row, one row, and, finally, one row; see Fig. 1.

The resulting matrix $A \in \mathcal{N}_W(R, S)$ is

$$A = \begin{bmatrix} 1.6 & 0.8 & 1.6 & 0 & 0 & 0 & 0 \\ 2.4 & 1.2 & 2.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 1.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 1.2 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 1.2 & 0.6 \end{bmatrix} \qquad \square$$

Lemma 6. *Assume that (4) holds and that these inequalities are strict for all $(p, q) \in C_W^1 \triangle C_W^2$. Then Algorithm A finds a matrix in $\mathcal{N}_W(R, S)$.*

Proof. It suffices to prove that a leading block in A may be found and that the updated data also satisfy the inequalities (4) strictly. The correctness of the algorithm will then follow by induction. So we may consider the first iteration (using W, R and S).

First, by Lemma 2 the structure of W is such that one of the three cases (i)–(iii) in Step 2 of Algorithm A will occur. We distinguish between these cases.

Consider first Case (i), when W is of type T_1 and $(p, q) \in C_W^2$. Since both r_i and s_j are strictly positive, $a_{ij} = r_i s_j/\tau > 0$ $(i \leqslant p,\ j \leqslant q)$. So the constructed leading q columns of A have the same pattern as the leading q columns of W. We calculate (as $\tau = \sum_{i \leqslant p} r_i$)

$$\sum_{i=1}^{m} a_{ij} = s_j \sum_{i=1}^{p} r_i/\tau = s_j \quad (j \leqslant q)$$

so the first q column sums of the matrix A are as desired. Moreover, since (4) holds with strict inequality for $(p, q) \in C_W^2$

$$\sum_{j=1}^{q} s_j < \sum_{i=1}^{p} r_i = \tau$$

and therefore

$$\sum_{j=1}^{q} a_{ij} = r_i \sum_{j=1}^{q} s_j/\tau < r_i \quad (i \leqslant p).$$

This implies that the updated row sum $\tilde{r} = r_i - \sum_{j=1}^{q} a_{ij} > 0$ as desired. Moreover, the updated $\widetilde{W}$ is a staircase matrix with at least one leading 1 in each of the first p rows (as W is of type T_1). Finally, we observe that the staircase support inequalities for $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{W}$ must hold. In fact, each such inequality may be obtained from an original staircase support inequality (for R, S and W) by subtracting $\sum_{j=1}^{q} s_j$ on each side of the inequality (due to our update of the first p row sums).

The proof in Case (ii) (when W is of type T_2) is similar (by interchanging the roles of rows and columns).

Finally, consider Case (iii). So $\widetilde{W} = J_{p,q} \oplus W'$ where $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$, and we define $\tau = \sum_{i \leqslant p} r_i = \sum_{j \leqslant q} s_j$ and $a_{ij} = r_i s_j/\tau$. Then A has the right row and column sums in the first p rows and the first q columns, and also its support there coincides with the support of W. If $p = m$ and $q = n$ we have determined A completely, otherwise the updated row and column sums, and $\widetilde{W}$ clearly satisfy the desired properties for the next iteration of the algorithm. This shows that Algorithm A correctly constructs a matrix A in $\mathscr{N}_W(R, S)$. $\square$

The proof of Theorem 5 now becomes very short.

Proof of Theorem 5. If $\mathscr{N}_W(R, S)$ is nonempty, then (4) holds due to Theorem 4, and these inequalities must be strict for all $(p, q) \in C_W^1 \triangle C_W^2$ (this follows from the summation in the beginning of the proof of Theorem 4).

Assume next that (4) holds with strict inequalities throughout. Then, by Lemma 6, Algorithm A may be used to construct a matrix in $\mathscr{N}_W(R, S)$ so this class is nonempty. $\square$

4. A connection to majorization

In this final section we discuss a connection between staircase transportation matrices and the notion of majorization. For further treatment of the theory of majorization and its applications, we refer to [8] and [4].

For nonincreasing vectors $a, b \in \mathbb{R}^n$ we say that a is *majorized* by b, and write $a \preceq b$, if $\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i$ $(k \leqslant n-1)$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. This notion reflects that the components of a are "less spread out" than the components of b.

Throughout this section W is the $n \times n$ staircase matrix where $v_i = i$ and $h_i = n$ $(i \leqslant n)$, so W has its ones on and above the main diagonal, and zeros elsewhere. We call W an *upper triangular pattern*. Then $C_W^1 = \emptyset$ and $C_W^2 = \{(k,k) : 1 \leqslant k \leqslant n-1\}$.

From Theorem 4 we obtain the following corollary.

Corollary 7. *Let R and S be nonincreasing, positive vectors of length n with $\sum_i r_i = \sum_j s_j$. Then the following are equivalent*:

(i) $S \preceq R$, *i.e.*, $\sum_{j=1}^k s_j \leqslant \sum_{j=1}^k r_j (1 \leqslant k \leqslant n-1)$.
(ii) *There is an upper triangular matrix in $\mathcal{N}(R,S)$.*
(iii) *The North-West Corner rule for $\mathcal{N}(R,S)$ produces an upper triangular matrix.*

In the situation of this corollary, the matrix A produced by Algorithm NWC says how to redistribute the "mass" represented by the components of R in order to obtain S. This is done by moving mass to components with higher index. Due to this interpretation we call each matrix in $\mathcal{N}_W(R,S)$ a *distribution matrix*.

Example 2. Let $R = (9,2,2,1,1)$ and $S = (5,3,3,2,2)$. Then Algorithm NWC produces the following upper triangular matrix:

$$A = \begin{bmatrix} 5 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $S \preceq R$ and the distribution matrix A shows how to obtain S from R by moving 3 and 1 from the first component to the second and third components, respectively, etc.

It is interesting to note that the distribution matrices for a given majorization $S \preceq R$ play a "similar role" as another matrix class associated with the majorization. The *majorization polytope* $\Omega_n(S \preceq R)$ consists of all doubly stochastic matrices M such that $S = MR$ where R and S here are considered as column vectors. This is a nonempty subpolytope of the Birkhoff polytope and it was discussed in detail in [3], see also [4]. While each matrix $M \in \Omega_n(S \preceq R)$ corresponds to a certain linear transformation that transforms R into S, the distribution matrices studied here give another interpretation of majorization as an "additive redistribution process".

Finally, we remark that, for other staircase patterns than upper triangular ones, the staircase support condition (4) may be seen as two majorization conditions (corresponding to C_W^1 and C_W^2) for appropriate partial sum vectors obtained from R and S. Thus nonemptyness of the face $\mathcal{N}_{\leqslant W}(R,S)$ of the transportation polytope may be expressed in terms of majorization-like inequalities.

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References

[1] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows: Theory, Algorithms, and Applications, Englewood Cliffs, New Jersey, Prentice-Hall, 1993.
[2] R.A. Brualdi, Convex sets of non-negative matrices, Canad. J. Math. 20 (1968) 144–157.
[3] R.A. Brualdi, The doubly stochastic matrices of a vector majorization, Linear Algebra Appl. 61 (1984) 141–154.
[4] R.A. Brualdi, Combinatorial Matrix Classes, Cambridge University Press, 2006.
[5] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[6] C.R. Johnson, D.P. Stanford, Patterns that allow given row and column sums, Linear Algebra Appl. 311 (2000) 97–105.
[7] D.G. Luenberger, Linear and Nonlinear Programming, second ed., Addison-Wesley Publishing Company, Reading, Massachusetts, 1984.
[8] A.W. Marshall, I. Olkin, Inequalities – Theory of Majorization and Its Applications, Academic Press, 1979.
[9] A. Schrijver, On the history of the transportation and maximum flow problems, Math. Program. 91 (3) (2002) 437–445.
[10] R. Vanderbei, Linear Programming: Foundations and Extensions, third ed., Springer, 2008.
[11] V.A. Yemelichev, M.M. Kovalev, M.K. Kravtsov, Polytopes, Graphs and Optimization, Cambridge University Press, Cambridge, 1984.